The "WhatPower" Function

Student Outcomes

- Students calculate a simple logarithm using the definition.

Lesson Notes

The term *logarithm* is foreign and can be intimidating, so we begin the lesson with a more intuitive function, the "WhatPower" function, which is a simple renaming of the logarithm function. Do not explain this function to students directly, but let them figure out what the function does. The first two exercises have already been solved to provide a hint of how the "WhatPower" function works.

This lesson is just the first introduction to logarithms, and the work done here will prepare students to solve exponential equations of the form $ab^{ct} = d$ (**F-LE.A.4**) and use logarithms to model relationships between two quantities (**F-BF.B.4a**) in later lessons. In the next lessons, students will create logarithm tables to discover some of the basic properties of logarithms before continuing on to look at the graphs of logarithmic functions, and then to finally modeling logarithmic data. In this lesson, we develop the ideas and notation of logarithmic expressions, leaving many ideas to be explored later in the module.

Classwork

Opening Exercise (12 minutes)

Allow students to work in pairs or small groups to complete these exercises. Do not explain this function to students directly, but allow them to struggle to figure out what this new "WhatPower" function means and how to evaluate these expressions. When there is about two minutes left, instruct groups that have not finished parts (a)–(p) to skip to part (q) so that all groups have time to think about and state the definition of this function. You may choose to collect the groups' definitions on paper and share some or all of them with the class using the document camera. We will work on refining this definition through the lessons; in particular, we are interested in the allowable values of the base b.

Opening Exercise

Evaluate each expression. The first two have been completed for you.

a. $\text{WhatPower}_2(8) = 3$

3, because $2^3 = 8$

b. $\text{WhatPower}_3(9) = 2$

2, because $3^2 = 9$

c. $\text{WhatPower}_6(36) =$ ______

2, because $6^2 = 36$

d. $\text{WhatPower}_2(32) =$ ______

5, *because* $2^5 = 32$

e. $\text{WhatPower}_{10}(1000) =$ ______

3, *because* $10^3 = 1000$

f. $\text{WhatPower}_{10}(1{,}000{,}000) =$ ______

6, *because* $10^6 = 1{,}000{,}000$

g. $\text{WhatPower}_{100}(1{,}000{,}000) =$ ______

3, *because* $100^3 = 1{,}000{,}000$

h. $\text{WhatPower}_4(64) =$ ______

3, *because* $4^3 = 64$

i. $\text{WhatPower}_2(64) =$ ______

6, *because* $2^6 = 64$

j. $\text{WhatPower}_9(3) =$ ______

$\frac{1}{2}$, *because* $9^{\frac{1}{2}} = 3$

k. $\text{WhatPower}_5(\sqrt{5}) =$ ______

$\frac{1}{2}$, *because* $5^{\frac{1}{2}} = \sqrt{5}$

l. $\text{WhatPower}_{\frac{1}{2}}\left(\frac{1}{8}\right) =$ ______

3, *because* $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

m. $\text{WhatPower}_{42}(1) =$ ______

0, *because* $42^0 = 1$

n. $\text{WhatPower}_{100}(0.01) =$ ______

-1, *because* $100^{-1} = 0.01$

o. $\text{WhatPower}_2\left(\frac{1}{4}\right) =$ ______

-2, *because* $2^{-2} = \frac{1}{4}$

p. $\text{WhatPower}_{\frac{1}{4}}(2) =$ ______

$-\frac{1}{2}$, *because* $\left(\frac{1}{4}\right)^{-\frac{1}{2}} = 4^{\frac{1}{2}} = 2$

MP.8

q. With your group members, write a definition for the function WhatPower_b, where b is a number.

The value of WhatPower_b is the number you need to raise b to in order to get x. That is, if $b^L = x$, then $L = \text{WhatPower}_b(x)$.

Discussion (3 minutes)

Discuss the definitions the students created, but do not settle on an official definition just yet. To reinforce the idea of how this function works, ask students a series of "WhatPower" questions, writing the expressions on the board or the document camera, and reading $\text{WhatPower}_b(x)$ as "What power of b is x?" Be sure that students are visually seeing the odd structure of this notation and hearing the question "What power of b is x?" to reinforce the meaning of this function that depends on both the parameter b and the variable x.

- $\text{WhatPower}_2(16)$
  - 4
- $\text{WhatPower}_2(4)$
  - 2
- $\text{WhatPower}_2(\sqrt{2})$
  - $\frac{1}{2}$
- $\text{WhatPower}_2(1)$
  - 0
- $\text{WhatPower}_2\left(\frac{1}{8}\right)$
  - -3

Exercises 1–9 (8 minutes)

The point of this set of exercises is for students to determine which real numbers b make sense as a base for the WhatPower_b function. Have students complete this exercise in pairs or small groups, and allow time for students to debate.

Exercises 1–9

Evaluate the following expressions and justify your answers.

1. $\text{WhatPower}_7(49)$

   $\text{WhatPower}_7(49) = 2$ *because* $7^2 = 49$.

2. $\text{WhatPower}_0(7)$

   $\text{WhatPower}_0(7)$ *does not make sense because there is no power of* 0 *that will produce* 7.

3. $\text{WhatPower}_5(1)$

   $\text{WhatPower}_5(1) = 0$ *because* $5^0 = 1$.

4. $\text{WhatPower}_1(5)$

   $\text{WhatPower}_1(5)$ *does not exist because for any exponent* L, $1^L = 1$, *so there is no power of* 1 *that will produce* 5.

5. $\text{WhatPower}_{-2}(16)$

   $\text{WhatPower}_{-2}(16) = 4$ *because* $(-2)^4 = 16$.

6. $\text{WhatPower}_{-2}(32)$

   $\text{WhatPower}_{-2}(32)$ *does not make sense because there is no power of* -2 *that will produce* 32.

7. $\text{WhatPower}_{\frac{1}{3}}(9)$

   $\text{WhatPower}_{\frac{1}{3}}(9) = -2$ *because* $\left(\frac{1}{3}\right)^{-2} = 9$.

8. $\text{WhatPower}_{-\frac{1}{3}}(27)$

   $\text{WhatPower}_{-\frac{1}{3}}(27)$ *does not make sense because there is no power of* $-\frac{1}{3}$ *that will produce* 27.

9. **Describe the allowable values of b in the expression $\text{WhatPower}_b(x)$. When can we define a function $f(x) = \text{WhatPower}_b(x)$? Explain how you know.**

   If $b = 0$ *or* $b = 1$, *then the expression* $\text{WhatPower}_b(x)$ *does not make sense. If* $b < 0$, *then the expression* $\text{WhatPower}_b(x)$ *makes sense for some values of* x *but not for others, so we cannot define a function* $f(x) = \text{WhatPower}_b(x)$ *if* $b < 0$. *Thus, we can define the function* $f(x) = \text{WhatPower}_b(x)$ *if* $b > 0$ *and* $b \neq 1$.

Discussion (5 Minutes)

Ask student groups to share their responses to Exercise 9, in which they determined which values of b are allowable in the WhatPower_b function. By the end of this Discussion, be sure that all groups understand that we need to restrict b so that either $0 < b < 1$ or $b > 1$. Then, continue on to rename the WhatPower function to its true name, the logarithm base b.

- What we are calling the "WhatPower" function is known by the mathematical term logarithm, built from the Greek word *logos* (pronounced lo-gohs), meaning ratio, and *arithmos* (pronounced uh-rith-mohs), meaning number. The number b is the base of the logarithm, and we denote the logarithm base b of x (which means the power to which we raise b to get x) by $\log_b(x)$. That is, whenever you see $\log_b(x)$, think of $\text{WhatPower}_b(x)$.

MP.8

Discuss the definition shown in the Frayer diagram below. Ask students to articulate the definition in their own words to a partner and then share some responses. Have students work with a partner to fill in the remaining parts of the diagram and then share responses as a class. Provide some sample examples and non-examples as needed to illustrate some of the characteristics of logarithms.

Logarithm

Definition of the logarithm base b

If three numbers L, b, and x with $0 < b < 1$ or $b > 1$ are related by $b^L = x$, then L is the logarithm base b of x, and we write $\log_b(x) = L$.

Characteristics

If $b > 1$ *and* $x > 1$, $\log_b(x) > 0$.
If $0 < b < 1$ *and* $x > 1$, *then* $\log_b(x) < 0$.

Examples

$\log_2(8) = 3$
$\log_5(1) = 0$
$\log_{10}(0.01) = -2$

Non-Examples

$\log_0(x)$
$\log_{-1}(x)$
$\log_{10}(-2)$

- What are some examples of logarithms?
  - $\log_2(4) = 2$
  - $\log_3(27) = 3$
  - $\log_{10}(0.10) = -1$
- What are some non-examples?
  - $\log_0(4)$
  - $\log_1(4)$
- Why can't $b = 0$? Why can't $b = 1$?
  - *If there is a number* L *so that* $\log_0(4) = L$*, then* $0^L = 4$*. But, there is no number* L *such that* 0^L *is* 4*, so this does not make sense. Similar reasoning can be applied to* $\log_1(4)$.
- Is $\log_5(25)$ a valid example?
  - *Yes.* $\log_5(25) = 2$ *because* $5^2 = 25$.
- Is $\log_5(-25)$ a valid example?
  - *No. There is no number* L *such that* $5^L = -25$*. It is impossible to raise a positive base to an exponent and get a negative value.*
- Is $\log_5(0)$ a valid example?
  - *No. There is no number* L *such that* $5^L = 0$*. It is impossible to raise a positive base to an exponent and get an answer of* 0.
- So what are some characteristics of logarithms?
  - *The base* b *must be a positive number not equal to* 1*. The input must also be a positive number. The output may be any real number (positive, negative, or* 0*).*

Examples (4 Minutes)

Lead the class through the computation of the following logarithms. These have all been computed in the Opening Exercise using the WhatPower terminology.

Examples

1. $\log_2(8) = 3$

   3, because $2^3 = 8$.

2. $\log_3(9) = 2$

   2, because $3^2 = 9$.

3. $\log_6(36) =$ ______

   2, because $6^2 = 36$.

4. $\log_2(32) =$ ______

   5, because $2^5 = 32$.

5. $\log_{10}(1000) =$ ______

   3, because $10^3 = 1000$.

6. $\log_{42}(1) =$ ______

   0, because $42^0 = 1$.

7. $\log_{100}(0.01) =$ ______

   −1, because $100^{-1} = 0.01$.

8. $\log_2\left(\frac{1}{4}\right) =$ ______

   −2, because $2^{-2} = \frac{1}{4}$.

Exercise 10 (6 minutes)

Have students complete this exercise alone or in pairs.

Exercise 10

9. Compute the value of each logarithm. Verify your answers using an exponential statement.

   a. $\log_2(32)$

      $\log_2(32) = 5$, *because* $2^5 = 32$.

   b. $\log_3(81)$

      $\log_3(81) = 4$, *because* $3^4 = 81$.

   c. $\log_9(81)$

      $\log_9(81) = 2$, *because* $9^2 = 81$.

   d. $\log_5(625)$

      $\log_5(625) = 4$, *because* $5^4 = 625$.

   e. $\log_{10}(1,000,000,000)$

      $\log_{10}(1,000,000,000) = 9$, *because* $10^9 = 1,000,000,00$.

   f. $\log_{1000}(1,000,000,000)$

      $\log_{1000}(1,000,000,000) = 3$, *because* $1000^3 = 1,000,000,000$.

   g. $\log_{13}(13)$

      $\log_{13}(13) = 1$, *because* $13^1 = 13$.

   h. $\log_{13}(1)$

      $\log_{13}(1) = 0$, *because* $13^0 = 1$.

Scaffolding:

- If students are struggling with notation, give them examples where they convert between logarithmic and exponential form.
- Use this chart as a visual support.

Logarithmic form	Exponential form
$\log_8(64) = 2$	
	$8^{-2} = \frac{1}{64}$
$\log_{64}(4) = \frac{1}{3}$	

i. $\log_9(27)$

$\log_9(27) = \frac{3}{2}$, *because* $9^{\frac{3}{2}} = 3^3 = 27$.

j. $\log_7(\sqrt{7})$

$\log_7(\sqrt{7}) = \frac{1}{2}$, *because* $7^{\frac{1}{2}} = \sqrt{7}$.

k. $\log_{\sqrt{7}}(7)$

$\log_{\sqrt{7}}(7) = 2$, *because* $(\sqrt{7})^2 = 7$.

l. $\log_{\sqrt{7}}\left(\frac{1}{49}\right)$

$\log_{\sqrt{7}}\left(\frac{1}{49}\right) = -4$, *because* $(\sqrt{7})^{-4} = \frac{1}{(\sqrt{7})^4} = \frac{1}{49}$.

m. $\log_x(x^2)$

$\log_x(x^2) = 2$, *because* $(x)^2 = x^2$.

Closing (2 minutes)

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.

Lesson Summary

- If three numbers, L, b, and x are related by $x = b^L$, then L is the *logarithm base* b of x, and we write $\log_b(x)$. That is, the value of the expression $L = \log_b(x)$ is the power of b needed to obtain x.
- Valid values of b as a base for a logarithm are $0 < b < 1$ and $b > 1$.

Exit Ticket (5 minutes)