Bacteria and Exponential Growth

Student Outcomes

- Students solve simple exponential equations numerically.

Lesson Notes

The lessons in Topic A familiarized students with the laws and properties of real-valued exponents. Topic B introduces the logarithm and develops logarithmic properties through exploration of logarithmic tables, primarily in base 10. This lesson introduces simple exponential equations whose solutions do not follow from equating exponential terms of equal bases. Because we have no sophisticated tools for solving exponential equations until we introduce logarithms in later lessons, we use numerical methods to approximate solutions to exponential equations, a process which asks students to determine a recursive process from a context to solve $2^x = 10$ (**F-BF.A.1a**, **F-BF.B.4a**, **A-CED.A.1**). Students will have many opportunities to solve such equations algebraically throughout the module, using both the technique of equating exponents of exponential expressions with the same base and logarithms. The goals of this lesson are to help students understand (1) why logarithms are useful by introducing a situation (i.e., solving $2^x = 10$) offering students no option other than numerical methods to solve it, (2) that it is often possible to solve equations numerically by trapping the solution through better and better approximation, and (3) that the better and better approximations are converging on a (possibly) irrational number.

Exponential equations are used frequently to model bacteria and population growth, and both of those scenarios occur in this lesson.

Classwork

Opening Exercise (6 minutes)

In this exercise, students work in groups to solve simple exponential equations that can be solved by rewriting the expressions on each side of the equation as a power of the same base and equating exponents. It is also possible for students to use a table of values to solve these problems numerically; either method is valid and both should be discussed at the end of the exercise. Asking students to solve equations of this type demands that they think deeply about the meaning of exponential expressions. Because students have not solved exponential equations previously, the exercises are scaffolded to begin very simply and progress in difficulty; the early exercises may be merely solved by inspection. When students are finished, ask for volunteers to share their solutions on the board and discuss different solution methods.

Opening Exercise

Work with your partner or group to solve each of the following equations for x.

a. $2^x = 2$

$x = 1$

b. $2^x = 2^3$

$x = 3$

Scaffolding:

Encourage struggling students to make a table of values of the powers of 2 to use as a reference for these exercises.

c. $2^x = 16$

$2^x = 2^4$
$x = 4$

d. $2^x - 64 = 0$

$2^x = 64$
$2^x = 2^6$
$x = 6$

e. $2^x - 1 = 0$

$2^x = 1$
$2^x = 2^0$
$x = 0$

f. $2^{3x} = 64$

$2^{3x} = 2^6$
$3x = 6$
$x = 2$

g. $2^{x+1} = 32$

$2^{x+1} = 2^5$
$x + 1 = 5$
$x = 4$

Scaffolding:

Give early finishers a more challenging equation where both bases need to be changed such as $4^{2x} = 8^{x+3}$.

Discussion (3 minutes)

This discussion should emphasize that the equations in the opening exercise have straightforward solutions because both sides can be expressed in terms of the common base 2 with an exponent.

MP.7

- How did the structure of the expressions in these equations allow you to solve them easily?
  - *Both sides of the equations could be written as exponential expressions with base* 2.
- Suppose the opening exercise had asked us to solve the equation $2^x = 10$ instead of the equation $2^x = 8$. Why is it far more difficult to solve the equation $2^x = 10$?
  - *We do not know how to express* 10 *as a power of* 2. *In the Opening Exercise, it is straightforward that* 8 *can be expressed as* 2^3.
- Can we find two integers that are over and under estimates of the solution to $2^x = 10$? That is, can we find a and b so that $a < x < b$?
  - *Yes; the unknown* x *is between* 3 *and* 4 *because* $2^3 < 10 < 2^4$.
- In the next example, we will use a calculator (or other technology) to find a more accurate estimate of the solutions to $2^x = 10$.

Example (12 minutes)

The purpose of this exercise is to numerically pinpoint the solution d to the equation $2^t = 10$ by squeezing the solution between numbers that get closer and closer together. We start with $3 < d < 4$, then find that $2^{3.3} < 10$ and $10 < 2^{3.4}$, so we must have $3.3 < d < 3.4$. Continuing with this logic, we squeeze $3.32 < d < 3.33$, and then $3.321 < d < 3.322$. The point of this exercise is that we can continue squeezing d between numbers with more and more digits, meaning that we have an approximation of d to greater and greater accuracy.

In the Student Materials, the tables for the Discussion below are presented next to each other, but they are spread out here so you can see how they fit into the discussion.

Example

The *Escherichia coli* bacteria (commonly known as *E. coli*), reproduces once every 30 minutes, meaning that a colony of *E. coli* can double every half hour. *Mycobacterium tuberculosis* has a generation time in the range of 12 to 16 hours. Researchers have found evidence that suggests certain bacteria populations living deep below the surface of the earth may grow at extremely slow rates, reproducing once every several thousand years. With this variation in bacterial growth rates, it is reasonable that we assume a 24-hour reproduction time for a hypothetical bacteria colony in the next example.

Suppose we have a bacteria colony that starts with 1 bacterium, and the population of bacteria doubles every day.

What function P can we use to model the bacteria population on day t?

$P(t) = 2^t$, for real numbers $t \geq 0$.

Have the students volunteer values of $P(t)$ to help you complete the following table.

t	$P(t)$
1	2
2	4
3	8
4	16
5	32

How many days will it take for the bacteria population to reach 8?

It will take 3 days, because $P(3) = 2^3 = 8$.

How many days will it take for the bacteria population to reach 16?

It will take 4 days, because $P(4) = 2^4 = 16$.

Roughly how long will it take for the population to reach 10?

Between 3 and 4 days; the number d so that $2^d = 10$.

We already know from our previous discussion that if $2^d = 10$, then $3 < d < 4$, and the table confirms that. At this point, we have an underestimate of 3 and an overestimate of 4 for d. How can we find better under and over estimates for d?

(Note to teacher: Once students respond, have them volunteer values to complete the table.)

Calculate the values of $2^{3.1}$, $2^{3.2}$, $2^{3.3}$, etc., until we find two consecutive values that have 10 between them.

t	$P(t)$
3.1	8.574
3.2	9.190
3.3	9.849
3.4	10.556

MP.8

From our table, we now know another set of under and over estimates for the number d that we seek. What are they?

We know d is between 3.3 and 3.4. That is, $3.3 < d < 3.4$.

Continue this process of "squeezing" the number d between two numbers until you are confident you know the value of d to two decimal places.

t	$P(t)$
3.31	9.918
3.32	9.987
3.33	10.056

t	$P(t)$
3.321	9.994
3.322	10.001

Since $3.321 < d < 3.322$, and both numbers round to 3.32, we can say that $d \approx 3.32$. We see that the population reaches 10 after 3.32 days, i.e., $2^{3.32} \approx 10$.

What if we had wanted to find d to 5 decimal places?

Keep squeezing d between under and over estimates until they agree to the first 5 decimal places. (Note to teacher: To 5 decimal places, $3.321928 < d < 3.321929$, so $d \approx 3.32193$.)

To the nearest minute, when does the population of bacteria become 10?

It takes 3.322 days, which is roughly 3 days, 7 hours and 43 minutes.

Discussion (2 minutes)

- Could we repeat the same process to find the time required for the bacteria population to reach 20 (or 100 or 500)?
  - *Yes, we could start by determining between which two integers the solution to the equation $2^t = 20$ must lie and then continue the same process to find the solution.*
- Could we achieve the same level of accuracy as we did in the example? Could we make our solution more accurate?
  - *Yes, we can continue to repeat the process and eventually "trap" the solution to as many decimal places as we would like.*

Lead students to the idea that for any positive number x, we can repeat the process above to approximate the exponent L so that $2^L = x$ to as many decimal places of accuracy as we would like. Likewise, we can approximate an exponent so that we can write the number x as a power of 10, or a power of 3, or a power of e, or a power of any positive number other than 1.

Note that there is a little bit of a theoretical hole here that will be filled in later when the logarithm function is introduced. In this lesson, we are only finding a rational approximation to the value of the exponent, which is the logarithm and is generally an irrational number. That is, in this example we are not truly writing 10 as a power of 2, but we are only finding a close approximation. If students question this subtle point, let them know that later in the module we will have definitive ways to write any positive number exactly as a power of the base.

Exercise (8 minutes)

Divide students into groups of 2 or 3, and assign each group a different equation to solve from the list below. Students should repeat the process of the Example to solve these equations by squeezing the solution between more and more precise under and over estimates. Record the solutions in a way that students can see the entire list either written on poster board, written on the whiteboard, or projected through the document camera.

Exercise

Use the method from the Example to approximate the solution to the equations below to two decimal places.

a. $2^x = 1000$ $\quad x \approx 9.97$

b. $3^x = 1000$ $\quad x \approx 6.29$

c. $4^x = 1000$ $\quad x \approx 4.98$

d. $5^x = 1000$ $\quad x \approx 4.29$

e. $6^x = 1000$ $\quad x \approx 3.85$

f. $7^x = 1000$ $\quad x \approx 3.55$

g. $8^x = 1000$ $\quad x \approx 3.32$

h. $9^x = 1000$ $\quad x \approx 3.14$

i. $11^x = 1000$ $\quad x \approx 2.88$

j. $12^x = 1000$ $\quad x \approx 2.78$

k. $13^x = 1000$ $\quad x \approx 2.69$

l. $14^x = 1000$ $\quad x \approx 2.62$

m. $15^x = 1000$ $\quad x \approx 2.55$

n. $16^x = 1000$ $\quad x \approx 2.49$

Discussion (2 minutes)

MP.3

- Do you observe a pattern in the solutions to the equations in Exercise 1?
  - *Yes, the larger the base, the smaller the solution.*
- Why would that be?
  - *The larger the base, the smaller the exponent needs to be in order to reach* 1000.

Closing (4 minutes)

Have students respond to the following questions individually in writing or orally with a partner.

- Explain when a simple exponential equation, such as those we have seen today, can be solved exactly using our current methods.
  - *If both sides of the equation can be written as exponential expressions with the same base, then the equation can be solved exactly.*
- When a simple exponential equation cannot be solved by hand, what can we do?
  - *Give crude under and over estimates for the solution using integers.*
  - *Use a calculator to find increasingly accurate over and under estimates to the solution until we are satisfied.*

Exit Ticket (8 minutes)